# Robust Eigensystem Assignment for Flexible Structures

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**An improved method is developed for eigenvalue and eigenvector placement of a closed-loop control system using either state or output feedback. The method basically consists of three steps. First, the singular value or QR decomposition is used to generate an orthonormal basis that spans admissible eigenvector space corresponding to each assigned eigenvalue. Second, given a unitary matrix, the eigenvector set that best approximates the given matrix in the least-square sense and still satisfies eigenvalue constraints is determined. Third, a unitary matrix is sought to minimize the error between the unitary matrix and the assignable eigenvector matrix. For use as the desired eigenvector set, two matrices, namely, the open-loop eigenvector matrix and its closest unitary matrix, are proposed. The latter matrix generally encourages both minimum conditioning and control gains. In addition, the algorithm is formulated in real arithmetic for efficient implementation. To illustrate the basic concepts, numerical examples are included.**

## Nomenclature

$A$ = open-loop state matrix ($2n \times 2n$)
$B$ = control input matrix ($2n \times m$)
$c_\kappa$ = real coefficients corresponding to $\kappa$th assignable eigenvector ($\nu_\kappa \times 1$)
$c_2(\ )$ = condition number of ( ) in matrix 2 norm
$\hat{c}_\kappa$ = optimal real coefficients corresponding to $\kappa$th assignable eigenvector ($\nu_\kappa \times 1$)
$G$ = feedback gain matrix ($m \times \gamma$)
$H$ = measurement matrix ($\gamma \times 2n$)
$m$ = number of inputs
$p$ = number of pairs of conjugate eigenvalues assigned
$Q$ = desired set of orthogonal eigenvectors ($2n \times 2p$)
$V_{o\kappa}$ = orthogonal basis for null space of $\Gamma_\kappa$
$V_{\sigma\kappa}$ = orthogonal complement to the null space of $\Gamma_\kappa$
$w_\kappa$ = weighting factor corresponding to least-square error of $\kappa$th eigenvector
$\Gamma_\kappa$ = expanded matrix characterizing the closed-loop system $4n \times (4n + 2m)$
$\gamma$ = number of outputs
$\lambda_{r\kappa}$, $\lambda_{i\kappa}$ = real and imaginary components of $\kappa$th closed-loop eigenvalue, respectively
$\nu_\kappa$ = dimension of the null space of $\Gamma_\kappa$
$\sigma$ = singular values
$\phi_{r\kappa}$, $\phi_{i\kappa}$ = real and imaginary components of assigned eigenvector corresponding to $c_\kappa$
$\Psi_{r\kappa}$, $\Psi_{i\kappa}$ = real and imaginary components of $\kappa$th closed-loop eigenvector, respectively
$|\ |_F$ = Frobenius norm

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## I. Introduction

It is known that eigensystem assignment using linear, constant state or output feedback for multi-input multioutput (MIMO) systems plays a key role in shaping transient response. Wonham[1] first related controllability and eigenvalue assignability for state feedback of MIMO systems. Issues arising from the nonuniqueness of control gains (or eigenvectors) for the placing eigenvalues were investigated by many researchers after Moore[2] characterized the class of all closed-loop eigenvector sets attainable for a given set of eigenvalues. Along a similar line, many coinciding results and extensions, such as multiple eigenvalue assignment for the output feedback, have been reported as exemplified by Refs. 3-5.

The problem of developing reliable algorithms for designing controllers that are robust in addition to satisfying eigenvalue placement constraints has recently been an area of active research. Robustness, as used here, refers to the property whereby the closed-loop eigenvalues are insensitive to the system uncertainties and/or perturbations. The class of perturbations considered here includes parameter perturbations in the plant or gain matrices for constant, linear, time-invariant dynamical systems. A recent review of robustness theory and analysis considered here is given in Ref. 6. Among the many published works related to robustness theory and optimization, there appear to be at least four methods that address the problem of robustness or sensitivity optimization subject to eigenvalue placement constraints. The first class of methods[7,8] involves the selection of eigenvector sets that simultaneously satisfy modal insensitivity and eigenvalue placement constraints. A major drawback to this approach is the difficulty in simultaneously satisfying modal insensitivity and eigenvalue placement constraint equations. Special cases are proposed that ease the difficulty. An important distinct feature in the first method is the need for evaluating derivatives of system matrices with respect to suspected uncertain parameters. The second class of methods[9,10] involve direct optimization of a scalar index that is related to the condition number while subject to eigenvalue placement constraints in the form of Sylvester's equations. Similarly to the latter approach, the third class of methods[6] considers the direct minimization of robustness measures, including a norm of eigenvalue sensitivity and various other robustness measures, subject to explicitly stated equality and inequality constraints on eigenvalues. The second and third methods involve iterative algorithms, which require computation of various sensitivities and may be computationally expensive for large systems. However, the use of sequential linear programming and homotopy methods[6,11] in the latter approach makes the algorithm numerically robust, easy to implement and, perhaps most importantly, informative when no solution exists for the stated problem. A fourth method[12] uses the concept of orthogonal projection into linear subspaces of eigenvectors to improve *iteratively* various equivalent measures of conditioning. This method utilizes an orthonormal basis spanning the subspaces that satisfy eigenvalue placement constraints. This method is based on QR and SVD (singular value decomposition) algorithms and is consequently numerically stable in addition to being efficient since no derivative calculations are required as in the previous methods.

In this paper, a design alorithm is introduced that amounts to an extension of the fourth method in at least three ways, namely: 1) it is noniterative 2) it extends to output feedback, and 3) it uses design freedom to encourage minimum gain configuration when the number of eigenvalues to be placed is less than the number assignable. Furthermore, several insights pertaining to the robust eigensystem assignment problem are presented. First, a closed-form expression for the set of eigenvectors that are closest, in a least-squares sense, to an a priori specified matrix and that still assign desired eigenvalues is obtained. The remaining freedom on the choice of an a priori matrix is discussed in detail. In particular, the unitary matrix closest to the open-loop eigenvector matrix is recommended. This choice is believed to produce both well-conditioning and minimum control effort. In addition, this paper addresses the important case in which fewer eigenvalues are specified than is possible. It is shown that the gain matrix satisfies an underdetermined set of linear equations, which naturally leads to a unique minimum norm gain. Furthermore, the need for minimum gain controllers motivates us to introduce a correlation criterion, which is based on orthogonal projections[13] for identifying the best desired eigenvector corresponding to each subspace of admissible eigenvectors. The correlation problem is clearly more evident in the direct output feedback case. The correlation procedure also takes into consideration the weighting of eigenvectors due to observability in addition to projection magnitudes. Finally, it should be mentioned that all formulations involve real arithmetic.

## II. State Feedback Formulation

In the analysis and design of the dynamics and vibration control of flexible structures, a set of second-order linear, constant coefficient, ordinary differential equations are frequently used. This leads to an even-dimensioned state space. Although not restricted to even dimensions, let $A$ be the state matrix of order $2n$ and $B$ the $2n \times m$ influence matrix of the $m$ control inputs for a linear, time-invariant, constant-gain feedback system. Assume that full state feedback is used to design the controller, with gain matrix $G$ of dimension $m \times 2n$. For computational efficiency, the eigensolution of the closed-loop system $[A + BG]$ can then be written as

$$[A + BG]\Psi_\kappa = \Psi_\kappa \Lambda_\kappa, \quad \kappa = 1,\ldots,n \tag{1}$$

where $\Psi_\kappa = [\Psi_{r\kappa}, \Psi_{i\kappa}]$ is the $2n \times 2$ real eigenvector matrix corresponding to the $2 \times 2$ real block eigenvalue matrix

$$\Lambda_\kappa = \begin{bmatrix} \lambda_{r\kappa} & \lambda_{i\kappa} \\ -\lambda_{i\kappa} & \lambda_{r\kappa} \end{bmatrix}$$

The subscripts $r$ and $i$ refer to the real and imaginary parts of the assumed self-conjugate set of eigenvectors whereas subscript $\kappa$ refers to the mode number. Expanding Eq. (1) yields the following two equations:

$$[A + BG]\Psi_{r\kappa} = \lambda_{r\kappa}\Psi_{r\kappa} - \lambda_{i\kappa}\Psi_{i\kappa} \tag{2}$$

$$[A + BG]\Psi_{i\kappa} = \lambda_{i\kappa}\Psi_{r\kappa} + \lambda_{r\kappa}\Psi_{i\kappa} \tag{3}$$

Rearranging Eqs. (2) and (3) in a compact matrix form produces

$$\begin{bmatrix} A - \lambda_{r\kappa} I & \lambda_{i\kappa} I & B & 0 \\ -\lambda_{i\kappa} I & A - \lambda_{r\kappa} I & 0 & B \end{bmatrix} \begin{bmatrix} \Psi_{r\kappa} \\ \Psi_{i\kappa} \\ G\Psi_{r\kappa} \\ G\Psi_{i\kappa} \end{bmatrix} \triangleq \Gamma_\kappa \phi_\kappa = 0 \tag{4}$$

Note that Eq. (4) also holds for the case of repeated complex conjugate pairs of eigenvalues. For the case of real, repeated eigenvalues, $\Gamma_\kappa$ and $\phi_\kappa^T$ in Eq. (4) reduce to $[A - \lambda I \,|B]$ and $[\Psi_\kappa^T, (G\Psi_\kappa)^T]$, respectively. The only assumption required in Eq. (4) is that the system be nondefective, i.e., that a full set of eigenvectors corresponding to the eigenvalues to be assigned must exist. Since the preceding generalizations are straightforward, only the case in which complex conjugate pairs of eigenvalues are assigned, which is most common in the vibration control of flexible structures, will be presented.

### Eigenvalue Assignment

To obtain the nontrivial solution space of the homogeneous equation (4), the singular value decomposition (SVD) is applied to the matrix $\Gamma_\kappa$, yielding

$$\Gamma_\kappa = U_\kappa \Sigma_\kappa V_\kappa^T = U_\kappa \begin{bmatrix} \sigma_\kappa & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{\sigma\kappa}^T \\ V_{o\kappa}^T \end{bmatrix} \tag{5}$$

It follows that the matrix $V_{o\kappa}$, represents a set of orthogonal basis vectors spanning the null space of the matrix $\Gamma_\kappa$, so that[13]

$$\Gamma_\kappa \phi_\kappa = \Gamma_\kappa V_{o\kappa} c_\kappa = 0 \tag{6}$$

Note that if $\Gamma_\kappa$ is not close to a matrix of lesser (or higher) rank (which is easily found from the singular values; hence the advantage of using SVD), the preceding basis for null space, $V_{o\kappa}$, can be computed more efficiently by taking the QR decomposition of $\Gamma_\kappa^T$.

To obtain an expression for gain matrix, choose a particular set of vectors, $\phi_\kappa$ ($\kappa = 1,\ldots,p$) satisfying Eq. (6), corresponding to some choice $c_\kappa$, and partition the vector $\phi_\kappa$ into four components such that

$$\Gamma_\kappa \phi_\kappa \triangleq \Gamma_\kappa \begin{bmatrix} \phi_{r\kappa} \\ \phi_{i\kappa} \\ \hat{\phi}_{r\kappa} \\ \hat{\phi}_{i\kappa} \end{bmatrix} = \Gamma_\kappa \begin{bmatrix} V_{or\kappa} \\ V_{oi\kappa} \\ \hat{V}_{or\kappa} \\ \hat{V}_{oi\kappa} \end{bmatrix} c_\kappa = 0, \quad \kappa = 1,\ldots,p \tag{7}$$

Observation of Eqs. (4) and (7) indicates that

$$G[\phi_{r\kappa}\ \phi_{i\kappa}] = [\hat{\phi}_{r\kappa}\ \hat{\phi}_{i\kappa}]; \quad \kappa = 1,\ldots,p \tag{8}$$

form the matrix equation

$$G\Phi \triangleq G[\phi_{r1}, \phi_{i1}, \phi_{r2}, \phi_{i2}, \ldots, \phi_{rp}, \phi_{ip}]$$
$$= [\hat{\phi}_{r1}, \hat{\phi}_{i1}, \hat{\phi}_{r2}, \hat{\phi}_{i2}, \ldots, \hat{\phi}_{rp}, \hat{\phi}_{ip}] = \hat{\Phi} \tag{9}$$

A matrix inversion is required in the computation of the gain matrix $G$. However, if the number of eigenvalues to be assigned, $p$, is less than the number assignable, $n$, Eq. (9) becomes underdetermined, which leads naturally to a minimum gain solution. In order to assure that Eq. (9) is well conditioned for inversion, the condition number of the matrix $\Phi$ should be the smallest possible. Interestingly, the numerical requirement for well-conditioning of the matrix inversion problem corresponds exactly to the eigenvalue conditioning problem because $\Phi$ consists of eigenvectors. In general, the set of eigenvectors in the matrix $\Phi$ generated by some choice of $c_\kappa$ [see Eq. (6)],

$$[V_{or1}, V_{oi1}, \ldots, V_{orp}, V_{oip}] \operatorname{diag}[c_1, c_1, \ldots, c_p, c_p]$$
$$\triangleq V_o c = \Phi = [\phi_{r1}, \phi_{i1}, \ldots, \phi_{rp}, \phi_{ip}] \tag{10}$$

will not be perfectly conditioned, i.e., orthogonal.

**Optimal Eigenvalue Conditioning**

From Eq. (10), the achievable (or admissible) eigenvectors corresponding to a set of eigenvalues can be rewritten as

$$\tilde{V}_{o\kappa} c_\kappa = \tilde{\phi}_\kappa, \qquad \kappa = 1, \ldots, p \tag{11}$$

where

$$\tilde{V}_{o\kappa} \triangleq \begin{bmatrix} V_{or\kappa} \\ V_{oi\kappa} \end{bmatrix}, \qquad \tilde{\phi}_\kappa \triangleq \begin{bmatrix} \phi_{r\kappa} \\ \phi_{i\kappa} \end{bmatrix}$$

and $c_\kappa$ is a coefficient vector of dimension

$$\nu_\kappa = 4n + 2m - \operatorname{rank}(\Gamma_\kappa) \tag{12}$$

Except for a small class of degenerate cases, $\Gamma_\kappa$ will usually be a full-rank matrix, so that the following relation holds:

$$\nu_\kappa = 2m \le 4n \tag{13}$$

It is clear then that Eq. (11) represents a set of overdetermined equations. Thus, eigenvectors cannot be arbitrarily assigned because of the insufficient number of independent control inputs. The number of control inputs consequently affects the degree of orthogonality or robustness achievable for the closed-loop system. The matrix $\Phi$ must therefore be constructed with some choices of $c_\kappa$, such that $\Phi$ is the best approximation to a perfectly conditioned matrix.

Based on the these observations, suppose that a unitary matrix $Q$, where

$$Q = [q_{r1}, q_{i1}, \ldots, q_{rp}, q_{ip}]$$

is given. The weighted sum error in the approximation of the unitary matrix by the achievable closed-loop eigenvector matrix is

$$J(Q, c_\kappa) = \sum_{\kappa=1}^{p} |\tilde{q}_\kappa - \tilde{V}_{o\kappa} c_\kappa|_F w_\kappa, \qquad \tilde{q}_\kappa = [q_{r\kappa}, q_{i\kappa}] \tag{14}$$

where $w_\kappa$ represents the weighting factor associated with the error in approximating the $\kappa$th desired vector $\tilde{q}_\kappa$. Assuming that $Q$ is fixed in Eq. (14), the least-squares solution for each mode that best approximates the unitary matrix $Q$ yields

$$\hat{c}_\kappa = (\tilde{V}_{o\kappa}^T \tilde{V}_{o\kappa})^{-1} \tilde{V}_{o\kappa}^T \tilde{q}_\kappa; \qquad \kappa = 1, \ldots, p$$
$$= [V_{or\kappa}^T V_{or\kappa} + V_{oi\kappa}^T V_{io\kappa}]^{-1} [V_{or\kappa}^T q_{r\kappa} + V_{oi\kappa}^T q_{i\kappa}] \tag{15}$$

To explore the possibility of improving the degree of orthogonality further, substitute Eq. (15) into Eq. (14) to obtain the following problem:

Minimize

$$J(Q)$$

Subject to

$$Q^T Q = I \tag{16}$$

where

$$J(Q) = \sum_{\kappa=1}^{p} \left| [I - \tilde{V}_{o\kappa} (\tilde{V}_{o\kappa}^T \tilde{V}_{o\kappa})^{-1} \tilde{V}_{o\kappa}^T] \tilde{q}_\kappa \right|_F w_\kappa$$

Equation (16) shows that the error cost function depends on the a priori chosen unitary matrix $Q$. An optimal unitary matrix that minimizes this error must exist. There are several ways to solve this equation numerically. However, a closed-form, perhaps approximate, solution is currently being sought.

In order to obtain a reasonably good solution, the unitary matrix closest to the open-loop eigenvector matrix is proposed as the desired set. Denoting the open-loop eigenvector matrix by $\Psi_o$, the unitary matrix $Q$ closest to the matrix $\Psi_0$ can be defined as[13]

Minimize

$$|\Psi_o - Q|_F$$

Subject to

$$Q^T Q = I \tag{17}$$

The closest unitary matrix $Q$ is then

$$Q = UV^T \tag{18}$$

where

$$\Psi_o = U \Sigma V^T$$

The reason for this particular choice is twofold. First, it is intuitively obvious that the control gains will be close to zero if the desired eigenvector is similar to the open-loop eigenvector when the desired eigenvalues are close to the open-loop eigenvalues. Second, the open-loop eigenvector may not be unitary, i.e., not perfectly conditioned, so that the closest unitary neighbor would encourage the selection of a well-conditioned achievable set.

**Discussion**

A comparison is made with results due to Kautsky et al.,[12] who pose the same basic problem, namely: find an orthonormal set of vectors that minimizes (maximizes) some weighted measure of subspace angles between the orthogonal set and the corresponding achievable (complementary to achievable) space. They propose an iterative algorithm involving plane rotations, to preserve orthogonality of a starting set, until acceptable values for various measures of conditioning are achieved. The difference between the method in Ref. 12 and the method presented in this paper is in the definition of the cost function for measuring nearness to orthogonality and the order of computations.

The choice of open-loop eigenvector and its closest unitary matrix, as proposed, is believed to be suitable for the purpose of generating a well-conditioned eigensystem with small control gains. The implication is that the element of iterative search (as suggested in Ref. 12) for the "optimal" unitary (or orthogonal) matrix appears unnecessary in practice for many test problems, as indicated in Ref. 16.

## III. Direct Output Feedback Formulation

It is generally agreed that full states are not usually available in practice without estimators or observers. In this section, the previous derivations for state feedback are generalized to design output feedback. The main characteristics of output feedback is its simplicity in generating control commands directly by a linear combination of the available outputs.

The eigenvalue problem can be written as

$$[A + BGH]\Psi_\kappa = \Psi_\kappa \Lambda_\kappa, \qquad \kappa = 1,\ldots,p \tag{19}$$

where $A$, $B$, $\Psi_\kappa$, and $\Lambda_\kappa$ are as previously defined in Eq. (1). Matrices $G$ and $H$ represent the output feedback gain and the measurement matrices of dimensions $m \times \gamma$ and $\gamma \times 2n$, respectively. The corresponding eigenvalue equations in real form are

$$\Gamma_\kappa \phi_\kappa = 0$$

where

$$\phi_\kappa = \begin{bmatrix} \Psi_{r\kappa} \\ \Psi_{i\kappa} \\ GH\,\Psi_{r\kappa} \\ GH\,\Psi_{i\kappa} \end{bmatrix}$$

and $\Gamma_\kappa$ is defined in Eq. (4). Since the $\Gamma_\kappa$ matrices are identical to the case for full state feedback, the corresponding admissible eigenvector spaces are identical. However, the procedures for computing the gain matrices differ. Note that the number of pairs of conjugate eigenvalues, $p$, that can be placed using output feedback by the above procedure is limited to

$$2p \le \max\,(m,\gamma) \tag{20}$$

The equation for computing the output feedback gain matrix for placing $p$ pairs of conjugate modes can be written as

$$GH[\phi_{r1},\phi_{i1},\ldots,\phi_{rp},\phi_{ip}] = [\hat{\phi}_{r1},\hat{\phi}_{i1},\ldots,\hat{\phi}_{rp},\hat{\phi}_{ip}]$$

or

$$GH\Phi = \hat{\Phi} \tag{21}$$

From Eq. (21), observe that the uniqueness of the gain matrix depends on the number of conjugate pairs of eigenvalues to be placed, i.e.,

$$2p < \gamma \qquad \text{underdetermined} \tag{22a}$$

$$2p = \gamma \qquad \text{unique} \tag{22b}$$

$$2p > \gamma \qquad \text{overdetermined} \tag{22c}$$

The previous conditions assume that the eigenvectors selected, $\Phi$, are completely observable. Equation (22a) is the most practical case in which the number of eigenvalue pairs to be placed are expected to be less than the number of independent measurements. The least-squares, minimum norm solution or the unique inverse solution can be expressed in terms of the Moore-Penrose pseudoinverse,[13]

$$G = \hat{\Phi}(H\Phi)^+ \tag{23}$$

where

$$(H\Phi)^+ = V_\sigma \Sigma_\sigma^+ U_\sigma^T$$

$$H\Phi = [U_\sigma | U_o] \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_k & \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} V_\sigma^T \\ V_o^T \end{bmatrix}$$

$$\Sigma_\sigma^+ = \text{diag}\,(1/\sigma_1,\ldots,1/\sigma_k)$$

The use of SVD provides reliable numerical results for the Moore-Penrose pseudoinverse in Eq. (23).[14] For the typical underdetermined case, Eq. (23) represents the minimum gain solution, which is highly desirable in practice. Indeed, the motivation for the work reported herein is to develop reliable algorithms for designing well-conditioned controllers with minimum gains.

Finally, for the case in which the number of actuators exceeds the number of measurements (which is uncommon in practice), similar analysis as described previously applies, and it will not be elaborated.

## IV. Maximum Projection Correlation Criterion

The freedom in selecting a particular unitary matrix to generate a corresponding set of admissible eigenvectors is further investigated in this section. In addition to robustness in the eigenvalue conditioning sense, physical limitations on controller magnitudes suggest, if possible, a minimum feedback gain configuration.

For a given set of orthogonal column vectors, the resulting condition number and the norm of gain matrix depend on the choice of the vector to be used for each mode. For problems with closely spaced modes, as is common in flexible space structures,[15] the correlation between open-loop modes and subspaces of admissible eigenvectors may not be obvious. This is also evident for unitary matrices, which do not have direct physical meaning. The approach taken here to resolve the above correlation problem is to search for the mode with the maximum orthogonal projection among the given set of desired eigenvectors into each admissible eigenvector subspace. Let

$$\rho_{rj}^\kappa = HV_{or\kappa}V_{or\kappa}^T q_{rj}, \qquad \rho_{ij}^\kappa = HV_{oi\kappa}V_{oi\kappa}^T q_{ij}$$

and $Q \triangleq [q_{r1}, q_{i1},\ldots, q_{rn}, q_{in}]$. Now, compute

$$\rho_{f(\kappa)} = \underset{j}{\text{Maximum}} \left|\rho_{rj}^\kappa\right| + \left|\rho_{ij}^\kappa\right|$$
$$\text{for } j = 1,\ldots,n,\ j \neq f(\gamma) \text{ and } \gamma = 1,\ldots,\kappa-1 \tag{24}$$

The integer $f(\kappa)$ ($\kappa = 1,\ldots,p$) is the desired eigenvector number having the maximum projection onto $\kappa$th subspace of admissible eigenvectors, and $\rho_{f(\kappa)}$ represents the maximum projection sum of real and imaginary components. The observability plays an important role in determining the correlation of desired eigenvectors with the subspaces of admissible closed-loop eigenvectors. Maximum projection is meaningless if the eigenvector is unobservable!

## V. Numerical Examples

**Example 1: Full State Feedback**

To demonstrate the method, consider a simple linear dynamical system consisting of three lumped mass-spring

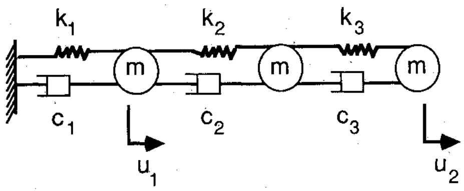


Fig. 1 Three-spring-mass-dashpot system.

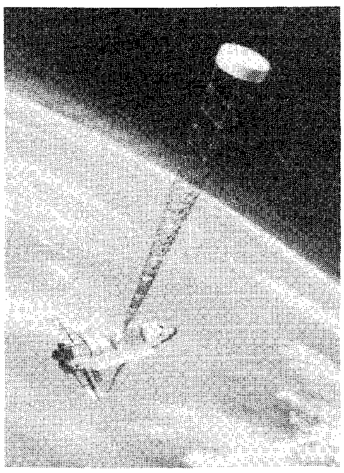

Fig. 2 Mast truss beam structure.

dashpots, connected in series and fixed at one end (see Fig. 1). The equation of motion can be written as

$$M\ddot{\xi} + C\dot{\xi} + K\xi = Du \qquad u = Gx \qquad x^T = [\xi^T, \dot{\xi}^T]$$

The mass, damping, stiffness, and force distribution matrices are, respectively, chosen as

$$M = \text{diag}\ (1,\ 1,\ 1)$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ & c_2 + c_3 & -c_3 \\ \text{Sym} & & c_3 \end{bmatrix}$$

$$K = \begin{bmatrix} 10 & -5 & 0 \\ & 25 & -20 \\ \text{Sym} & & 20 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrices represent a system with two actuators located at degrees of freedom 1 and 3.

*Example 1.1: Full State Feedback with No Damping*

The undamped system has $c_1 = c_2 = c_3 = 0$. The frequency and damping ratios are given in Table 1, where the eigenvalue $\lambda_i = -\zeta_i\omega \pm j\omega_i(1 - \zeta_i^2)^{1/2}$. The condition number of the corresponding undamped open-loop modal matrix is unity because the eigenvectors are orthogonal. The norm of gains and condition numbers corresponding to four sets of desired eigenvectors are given in Table 2.

Table 1 Frequency and damping ratio for the mass-spring example

| | Open-loop | | Closed-loop | |
|---|---|---|---|---|
| Mode no. | $\omega^o$ | $\zeta^o$ | $\omega$ | $\zeta$ |
| 1 | 1.0344 | 0.0 | 1.0482 | 0.5765 |
| 2 | 3.2934 | 0.0 | 3.2891 | 0.2959 |
| 3 | 6.5638 | 0.0 | 6.4857 | 0.0651 |

Table 2 Condition number and norm of gain

| Case | $Q$ | Maximum projection correlation | $\|\|G\|\|_F$ | $c_2(\Psi)$ |
|---|---|---|---|---|
| 1 | I | No | 38.23 | 11.24 |
| 2 | I | Yes | 21.81 | 9.05 |
| 3 | $\Psi_o$ | No | 2.85 | 3.17 |
| 4 | $\Psi_o$ | Yes | 2.85 | 3.17 |
| 5 | (Direct velocity feedback) | | 2.82 | 6.96 |

Case 5 in Table 2 represents the directly collocated velocity feedback configuration of the form

$$u_1 = -2\dot{x}_1 \qquad u_3 = -2\dot{x}_3$$

From Table 2, it is seen that the use of open-loop eigenvectors as desired closed-loop eigenvectors gives the smallest condition numbers and small gains. The correlation of a desired set of eigenvectors with respect to the assignable eigenvector subspaces improved the gain norm and condition number for the identity matrix. However, for the open-loop eigenvector case, the average correlation using eigenvalue magnitudes is sufficiently good (0.9488 out of a perfect correlation of 1) such that the eigenvector correlation is not needed.

The gain matrices for cases 3 and 4 are as follows:

$$G = \begin{bmatrix} -0.0057 & -0.1199 & 0.1124 & -1.8815 & 0.1687 & 0.0299 \\ -0.1985 & 0.1200 & -0.0126 & 0.1340 & -0.1001 & -2.1185 \end{bmatrix}$$

This gain matrix is close to the "physically robust" directly collocated velocity feedback configuration, as mentioned earlier. It is interesting to note that the open-loop eigenvector case gives a more robust (in the eigenvalue conditioning sense) closed-loop configuration than the collocated velocity feedback configuration for this example. Of course, the eigenvalue/eigenvector assignment algorithm is an "inverse" design algorithm, whereas the direct velocity feedback case is obtained by a "forward" design process in which eigenvalues cannot be assigned conveniently.

*Example 1.2: Full State Feedback with Damping*

The damping coefficients used were $c_1 = 2$, $c_2 = 0.5$, and $c_3 = 2$. The frequency and damping ratios are given in Table 3. It is seen that the open-loop eigenvalues and closed-loop eigenvalues are not well correlated.

Table 3 Frequency and damping ratio for the mass-spring-dashpot example

| Mode no. | Open-loop $\omega^o$ | Open-loop $\zeta^o$ | Closed-loop $\omega$ | Closed-loop $\zeta$ |
|---|---|---|---|---|
| 1 | 1.0481 | 0.1442 | 3.1622 | 0.3162 |
| 2 | 3.2517 | 0.3647 | 3.8078 | 0.3939 |
| 3 | 6.5606 | 0.3296 | 4.4384 | 0.3830 |

Table 4 Condition number and norm of gain

| Case | $Q$ | Maximum projection correlation | $\|\|G\|\|_F$ | $c_2(\Psi)$ |
|---|---|---|---|---|
| 1 | I | No | 29.76 | 28.72 |
| 2 | I | Yes | 27.62 | 22.97 |
| 3 | $\Psi_o$ | No | 26.96 | 26.20 |
| 4 | $\Psi_o$ | Yes | 26.77 | 22.41 |
| 5 | $UV^T$ | No | 26.88 | 23.11 |
| 6 | $UV^T$ | Yes | 26.81 | 21.72 |

The condition number of the open-loop modal matrix is 6.5249 and not unity because the eigenvectors are not orthogonal. The norm of gains and condition numbers corresponding to various sets of desired eigenvectors are shown in Table 4.

From the norm of gain matrices, it is seen that the choice of closed-loop eigenvectors closest to identity matrix (case 1) requires a large control effort with the worst conditioning. With projection correlation, the gains and condition number are improved (case 2). The use of open-loop eigenvectors as desired closed-loop eigenvectors without (case 3) and with (case 4) projection correlation gives improved results over the case of identity matrices. With the use of the closest unitary matrix to the open-loop eigenvector matrix (cases 5 and 6), a slight further improvement results. The closest unitary matrix, as expected, gives a smaller condition number than the corresponding nonorthogonal open-loop eigenvector matrix.

Although no theoretical basis exists at present, numerical results reported in Ref. 16 showed that, among a large set of test unitary matrices generated at random, the case closest to the open-loop eigenvector matrix produced a norm of gain matrix and condition number close to the optimum values of this set. Further work is needed to find the global optimum, noniteratively if possible, for Eq. (16).

*Example 2: Direct Output Feedback*

The system investigated is a reduced-order finite-element model of the MAST truss beam structure as shown in Fig. 2 (see Refs. 17 and 18 for detailed descriptions). The model includes the deployer-retractor assembly, Shuttle inertia properties, and rigid platforms for sensors and actuators allocation. There are three secondary actuator locations distributed along the beam and one primary at the tip. Each of the secondary actuator stations contains two actuators acting in the same plane. The primary station has four actuators to impart torques as well as in-plane forces. Also included are displacement and velocity sensors collocated with the actuators for a total of 20 measurements.

The reduced-order model consists of 92 first-order equations, and it includes actuator/sensor dynamics and 6 rigid body degrees of freedom in addition to elastic deformation. A total of 10 actuators using 20 output measurements (out of which 19 were found to be linearly independent) are used to increase the damping ratio of a selected set of elastic modes to 5%. Although 19 eigenvalues are assignable, only 9 pairs of complex eigenvalues (corresponding to elastic modes) are assigned. The corresponding frequencies and damping values are shown in Table 5.

Table 5 Frequency and damping ratio for MAST structure

| Mode no. | Open loop $\omega^o$, Hz | Open loop $\zeta^o$ | Closed loop $\omega$, Hz | Closed loop $\zeta$ |
|---|---|---|---|---|
| 1 | 0.1833 | 0.0044 | 0.1833 | 0.0500 |
| 2 | 0.2411 | 0.0049 | 0.2411 | 0.0500 |
| 3 | 1.3308 | 0.0232 | 1.3308 | 0.0500 |
| 4 | 1.3876 | 0.0223 | 1.3876 | 0.0500 |
| 5 | 1.5541 | 0.0211 | 1.5541 | 0.0500 |
| 6 | 3.7919 | 0.0063 | 3.7919 | 0.0500 |
| 7 | 3.9698 | 0.0061 | 3.9698 | 0.0500 |
| 8 | 5.3483 | 0.0051 | 5.3483 | 0.0500 |
| 9 | 6.6509 | 0.0054 | 6.6509 | 0.0500 |

Since not all eigenvalues can be assigned, because of the limitation of the number of measurements or the actuator, the remaining eigenvalues are not guaranteed to remain stable. For this example, two residual modes became unstable. Further research is needed to alleviate the stability problem for direct output feedback.

## VI. Concluding Remarks

In this paper, a novel algorithm using real arithmetic is presented for eigenvalue placement using singular value decomposition (SVD) for state or output feedback. By using the orthogonal basis generated by SVD to span the subspace of eigenvectors satisfying desired eigenvalue, the optimal set of eigenvectors can be written down easily, provided the optimum unitary matrix is known a priori. For cases in which the number of eigenvalues to be assigned is less than the order of the system, low eigenvalue sensitivity and small control gains can be obtained by carefully selecting the desired eigenvector set via the orthogonal projection correlation criteria. Numerical examples provide some indication of the simplicity and usefulness of the approach and, in particular, the advantage of using the open-loop eigenvector matrix or its closest unitary matrix as the desired eigenvector matrix. However, it is clear that the results outlined here are by no means complete and further work is needed to address the stability problem related to the unassigned eigenvalues for the direct output feedback.

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